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Research Article

Algebraic models for the aggregation of area features based upon morphological operators

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Abstract. Generalization is a fundamental function in GIS. It has been an important research theme for many years in cartography and GIS. A number of generalization operations have been identified, however most of them, especially those rule-based operations, remain at the conceptual level. This paper describes a set of mathematical (algebraic) models for area aggregation based on the operators developed in mathematical morphology. In this paper, the process of area aggregation is decomposed into two components, viz., combination and shape refinement, and algebraic models for both components are developed. These are demonstrated using various examples. The models provide a mathematical basis for area aggregation in digital generalization of map and other spatial data. The results show that these algebraic models have the potential for successful application.

1. Introduction

As pointed out by Abler (1987) and Müller *et al.* (1995), generalization is not only a cartographic process but also a fundamental function in spatial data handling and thus for GIS. In this digital era, generalization has become increasingly important since it is needed whenever multi-scale problems of spatial data are considered. Indeed, generalization has nowadays become part of the international research agenda in the spatial information sciences (Marble 1984, Abler 1987, Rhind 1988, Müller 1991). Over the past three decades, many projects have been initiated world-wide and a great number of papers on this topic have been published. However, many fundamental problems remain unresolved.

From the literature, it can be seen that a few strategies for generalization have been developed (e.g., Brassel and Weibel 1988, Shea and McMaster 1989) and a number of generalization operations (such as selection, omission, aggregation, coarsen, collapse and displacement) have been identified by researchers (e.g., Rhind 1973, Keates 1989, Shea and McMaster 1989, Beard and Mackaness 1991). However, the current situation is that many of these operations remain at the conceptual level and there is an urgent need to develop many still 'missing' algorithms or mathematical models for various generalization operations. Indeed, this goal has recently been prioritized by the ICA Working Group on Automated Map Generalization, (Weibel 1995).

This paper describes a set of mathematical models for one of the operations, namely, area aggregation in the geometric context. It aims only to describe a set of mathematical models, which may be used for transforming spatial representation from one scale to another (smaller) scale. Such models might be compared to other transformation models, such as affine and conformal models.

In terms of relevant methodology, it is notable that most research into automated map generalization is currently focused on vector data, even for area features, with some exceptions (e.g., Monmonier 1983, Jäger 1990, Schylberg 1993). However, it should be more convenient to manipulate area features in raster mode since raster is a space-primary data structure. This study will concentrate on raster data, in other words, the mathematical models developed in this study are in raster mode. More precisely, they are built upon the operators developed in mathematical morphology, which is a science of shape, form and structure.

Following this introduction is a discussion of the strategy for the aggregation of area features. Then, various operators developed in mathematical morphology are briefly introduced in order to provide a mathematical background. After that, mathematical models for area aggregation, which are built upon the basic morphological operators, are described in detail with a number of examples and, at the same time, the relationship between the scale of spatial data and the size of the structuring element in a morphological operator is discussed.

2. Area aggregation: problems and strategy

It can be noted from the literature that most research efforts have been spent on line generalization and much less investigation into the generalization of area features has been carried out. Therefore, it seems pertinent to give a brief review of generalization operations for area features before the aggregation problem is discussed in detail.

2.1. *Generalization operations for area features*

The literature shows that many operations for the generalization of area features, such as selection, elimination (or selective omission), aggregation, combination, collapse, coarsen, etc, have already been identified by researchers (e.g., Keates 1989, Beard and Mackaness 1991, McMaster and Shea 1992). These operations are illustrated in figures 1 to 4.

Some of these operations are essentially the same but use different terminology. For example, combination used by Keates (1989) is in fact the same as aggregation suggested by McMaster and Shea (1992). Recently, some research into collapse and aggregation has been carried out (Chithambaram *et al.* 1991, Monmonier 1983, Schylberg 1993). Area-patch generalization, which involves selective omission and

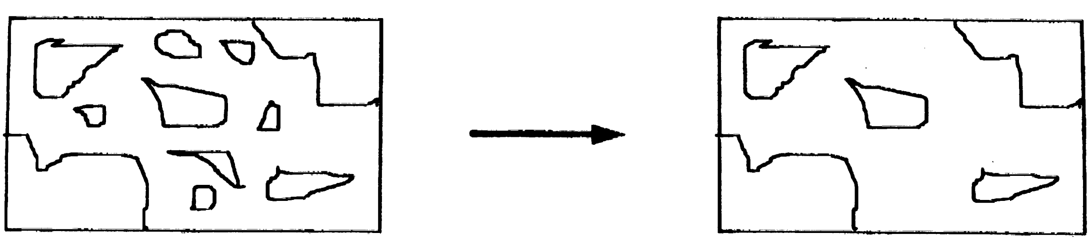
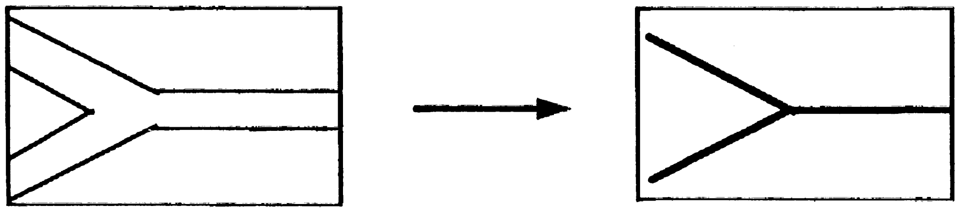


Figure 1. Selection and elimination operations for feature areas.

(*a*)



(*b*)

Figure 2. Collapse operation for area features. (*a*) Area to point, (*b*) Area to line.

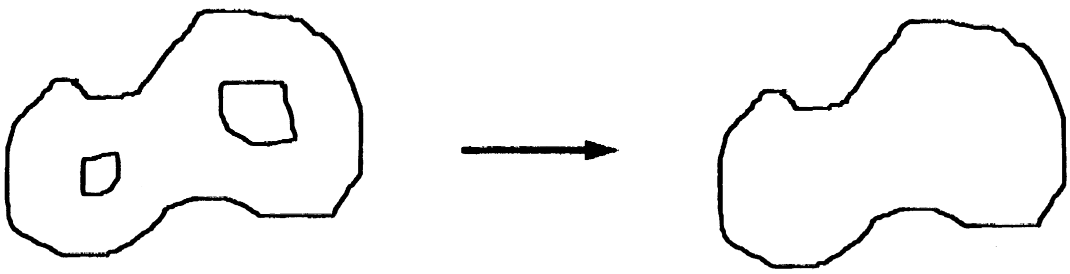


Figure 3. Coursen operation for area features.

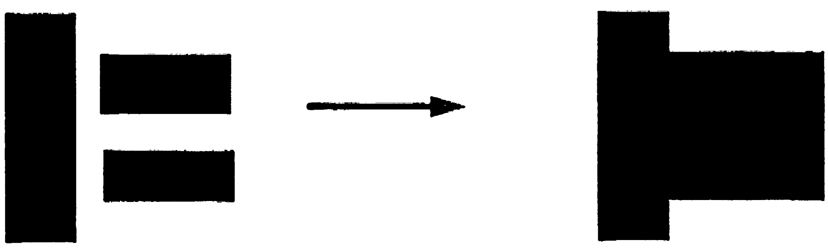


Figure 4. Aggregation operation for area features.

aggregation, has also been investigated (Müller and Wang 1992, Su and Li 1995). This paper deals with only one of these operations for area features, viz., area aggregation. The main purpose of this study is to provide some techniques which are mathematically more elegant.

2.2. *Area aggregation: a two-step process*

By a closer examination, the authors find that the aggregation process can be decomposed into two components (or two steps),

(*a*) Natural combination, and
(*b*) Shape refinement.

The first step combines those area features which are so close that they are not visually separable after a scale reduction (hence the term 'natural' being used). The critical value of the closeness is called *threshold of separation.* This combination process should happen naturally with scale reduction, albeit uncontrolled, as will be shown later. Also, after combining a number of small features, it is possible that the boundary of the resulting feature becomes so very irregular that a simplification process is needed. This simplification process is called *shape refinement* here. Two methods for shape refinement are proposed, viz convex hull and irregularity filtering (see later).

Both area combination and shape refinement will be discussed in detail and algebraic models for the two step processes will also be described. These models are in raster mode and built upon the operators developed in mathematical morphology.

3. Mathematical background: operators in mathematical morphology

Using mathematical morphology to build mathematical models for generalization of spatial data means to build models for these operations upon the two basic operators developed in mathematical morphology, i.e., dilation and erosion. These can be compared to +, −, × and ÷ in ordinary algebra. In order to facilitate the discussion of the mathematical models developed by the authors, the basic concepts in mathematical morphology are introduced here.

3.1. *Two basic operators in mathematical morphology*

Mathematical morphology is a science of form and structure, based on set theory. It was developed by French geostatistical scientists G. Matheron and J. Serra in the 1960s (Matheron 1975, Serra 1982). Since then it has found increasing application in digital image processing. Efforts have also been made by researchers to apply morphological tools to map generalization (Li 1994, Li and Su 1995, Su and Li 1995) and mapping related sciences, such as digital terrain modelling (Li and Chen 1991). The two basic operators are defined as follows (see Serra 1982, Haralick *et al.* 1987):

$$\text{Dilation:} \quad A \oplus B = \{a+b: a \in A, b \in B\} = \cup_{b \in B} A_b \tag{1}$$

$$\text{Erosion:} \quad A \ominus B = \{a: a+b \in A, b \in B\} = \cap_{b \in B} A_b \tag{2}$$

where A is the image to be processed and B is called the structuring element. In equation (1), it is called 'dilation of A by B' and in equation (2) 'erosion of A by B'. Examples of these two operators are given in figure 5, where the features are represented by black pixels. (The origin of a structuring element is considered to be its geometric centre if there is no other specific indication).

3.2. *Structuring elements*

The structuring element is a critical element in any morphological operation and it can be compared to the kernel (or mask) in a convolution operation. Indeed, it is through the convolution with the structuring element that a morphological operator changes the shape of the original image (or object). A structuring element can take any shape (square, cross) and size (e.g., 2 × 2 or 3 × 3). Figure 6 shows some of the possible shapes, i.e., circular, diagonal, linear, square and cross.

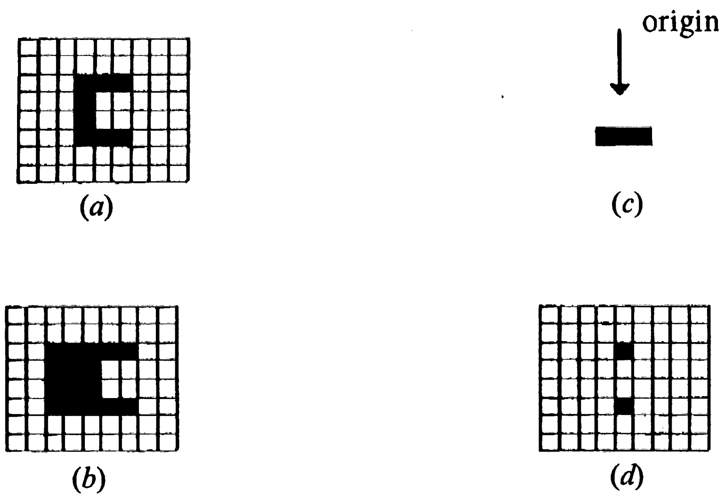


Figure 5. Two basic morphological operators: dilation and erosion. (*a*) Original image A, (*b*) The structuring element B, (*c*) A dilated by B (A⊕B), (*d*) A eroded by B (AΘB).

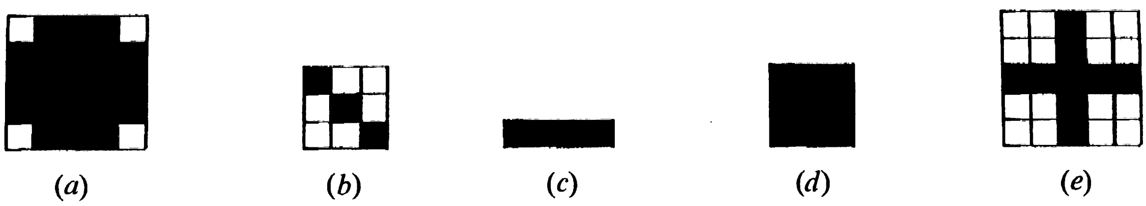


Figure 6. Some possible structuring elements. (*a*) Circle, (*b*) Diagonal, (*c*) Linear, (*d*) Square, (*e*) Cross.

Here, 'circular' means that it is used to approximate a circle in discrete raster metrics.

3.3. *Other morphological operators*

If a symmetric structuring element such as those shown in figure 6 (*d*) or figure 7 (*b*) is used for dilation, then the shape of the original image will be expanded uniformly along all directions. The dilation in this particular case is called expansion. Similarly, the erosion in this case is called shrink. These two special operations are illustrated

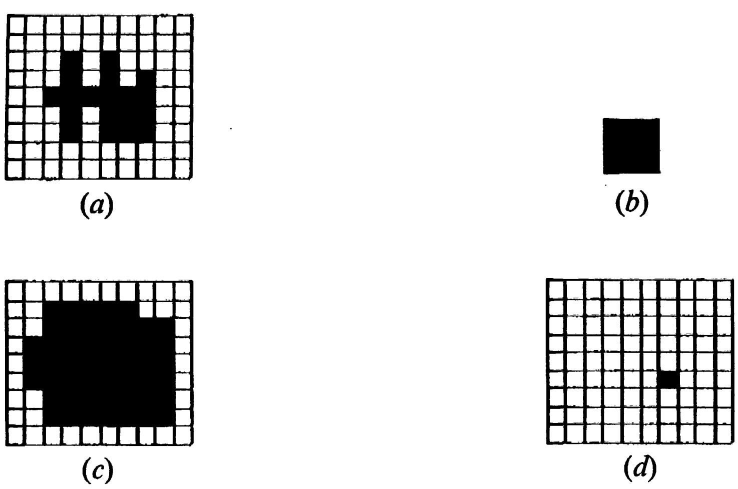


Figure 7. Expansion and shrink: special cases of dilation and erosion. (*a*) The original image A, (*b*) The structuring element B, (*c*) Expanded image C = A⊕B. (*d*) Shrunk image D = AΘB.

in figure 7 (*c*) and figure 7 (*d*). Therefore, the expansion and shrink operations used in ordinary image processing techniques are simply special cases of dilation and erosion.

Based on these two basic operators, i.e., dilation and erosion, a number of other operators have also been developed, such as closing, opening, thinning, thickening, hit or miss, conditional dilation, conditional erosion, conditional thinning, conditional thickening, sequential dilation, conditional sequential dilation, and so on (see Serra 1982, Haralick *et al.* 1987). Among them, the *opening* and *closing* operators are very suitable for the manipulation of area features. These two operators are defined as follows:

$$\text{Open:}\quad A \circ B = (A \ominus B) \oplus B \tag{3}$$

$$\text{Close:}\quad A \bullet B = (A \oplus B) \ominus B \tag{4}$$

where A is the original feature and B is the structuring element. Examples of these two operators are given in figure 8.

4. Algebraic models for natural combination of area features

As has been discussed in § 2.2, two steps are involved in the aggregation process, i.e. natural combination and shape refinement. This section will discuss the first step.

4.1. *A general model*

Before discussing in detail the algebraic models for feature combination, it is appropriate to discuss the general form of algebraic models. In general, an algebraic

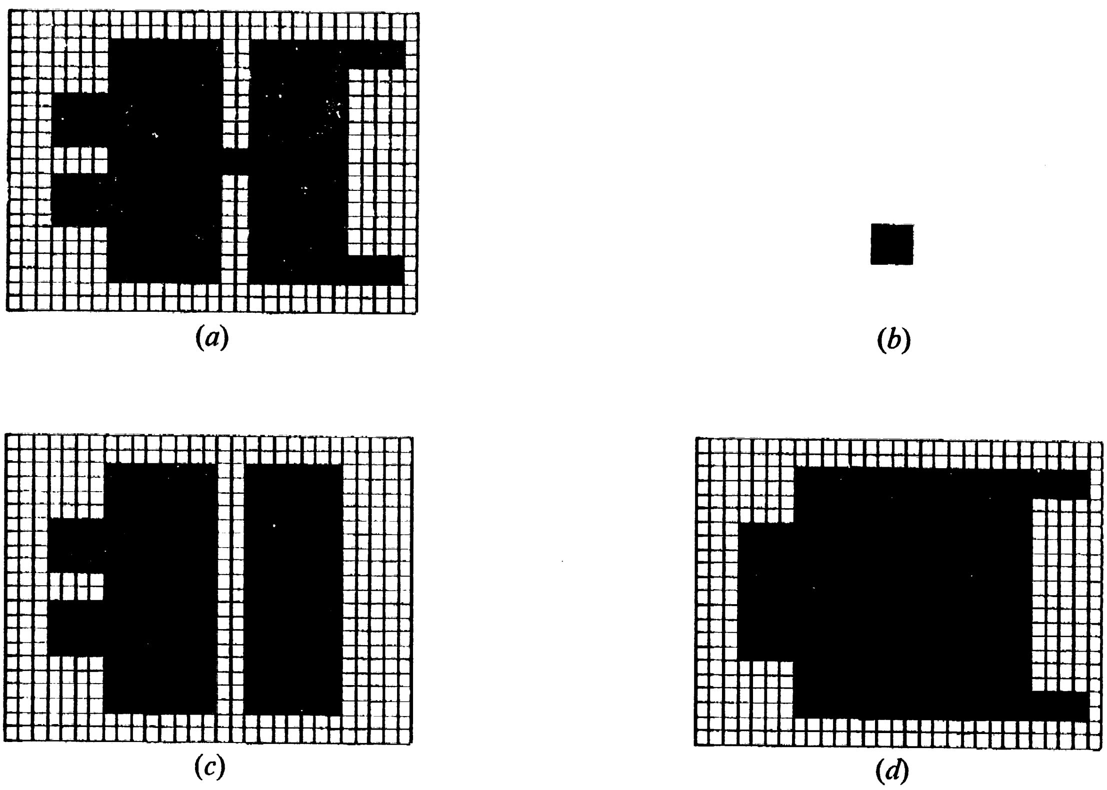


Figure 8. Opening and closing operators. (*a*) Original feature A, (*b*) The structuring element, (*c*) A is opened by B: A○B, (*d*) A is closed by B: A●B.

model based on morphological operators can be expressed as follows:

$$T = G_1(B_1)G_2(B_2)\ldots G_n(B_n) \tag{5}$$

where G_i is a morphological operator and B_i is a structuring element. The key is to determine the structuring element B_i used in equation (5). Each structuring element B_i can be described by two basic parameters, i.e., shape and size. These can be represented by some mathematical expressions. The formula for the size of a structuring element can be written as follows:

$$B_{size} = F(S_s, S_t) \tag{6}$$

where F is a function which governs the size of the structuring element, $1:S_s$ is the source scale and $1:S_t$ is the target scale of the spatial data; B_{size} is the size of the structuring element, which means the maximum pixel number crossing the origin of the structuring element. For example, the sizes of the structuring elements shown in figure 6 are 5, 3, 4, 3 and 5 respectively. Similarly, the formula for the shape of a structuring element can be written as follows:

$$B_{shape} = H(O_{shape}) \tag{7}$$

where O_{shape} is the shape of original feature; and H is a function to determine the shape of the structuring element according to O_{shape}.

4.2. *An algebraic model for area combination*

To develop an algebraic model for area combination means to make equation (5) specifically for combination purposes. Through an analysis of the characteristics of area combination, the following simple algebraic model is suggested:

$$C = (A \oplus B_1) \ominus B_2 \tag{8}$$

where A is the image showing the original features and B_1 and B_2 are the two structuring elements. When $B_1 = B_2$, equation (8) becomes the closing operator. The success of applying this model to area combination depends on the proper size and shape of structuring elements B_1 and B_2. A discussion of how to determine these two parameters will be given in the next two sub-sections.

4.3. *Determining the size of structuring elements for the combination model*

To determine the appropriate size of a structuring element, scale is the main factor to be considered. In other words, the size of the structuring element is dependent on the source scale and the target scale of the spatial data. This is obvious. When the scale is reduced N times, the space between two map features in terms of map distance will also be reduced N times. Therefore, it can be reasoned that the appropriate value of the structuring elements in equation (8) can be calculated as follows:

$$B_{size} = \frac{S_{target}}{S_{source}} \times D_s \tag{9}$$

where $1:S_{source}$ and $1:S_{target}$ are the scales of the source and target data, respectively. D_s is the distance at source scale in terms of the number of pixels below which two objects on the source map cannot be further separated. This value is the *threshold of separation* which is approximately 0·2 mm in terms of map distance. B_{size} is the size of the structuring element in terms of the number of pixels at target scale.

If a symmetric structuring element with the origin at its centre is to be used, then the dimension of the structuring element should be an odd number. In this case, equation (9) can be written as follows:

$$B_{size} = INT\left(\frac{INT\left(\dfrac{S_{target}}{S_{source}} \times D_s + 0{\cdot}5\right)}{2}\right) \times 2 + 1 \tag{10}$$

where INT means the integer part of the value. Figure 9 shows some structuring elements of different sizes, which can be used for various scale reductions. Figure 10 demonstrates the transformation process for area combination based on equation (8) and equation (10). In this particular case, figure 10(*a*) shows a group of area features on the original map with scale 1 : *S*. Figure 10(*b*1), (*c*1), (*d*1) and (*e*1) are the results obtained by applying equation (8) to the original features with the structuring element shown in figure 9 (*a*), (*b*), (*c*) and (*d*) respectively. After this process, the data are combined to suit the representation at the target scale.

4.4. *Determining the shape of structuring elements for the combination model*

The global and general shape of area features need to be kept after the combination operation. To do so, the shape of the structuring element should be kept in accordance with the original shape of the area features. Shapes such as circle, square, line and diagonal are among the possible choices of structuring elements for area combination. In general, it is suggested that rectangular structuring elements be used for rectangular features and circular structuring elements be used for curved features (i.e., with natural shapes). The examples shown in figures 11 to 17 show the combination of various area features with different types of structuring elements.

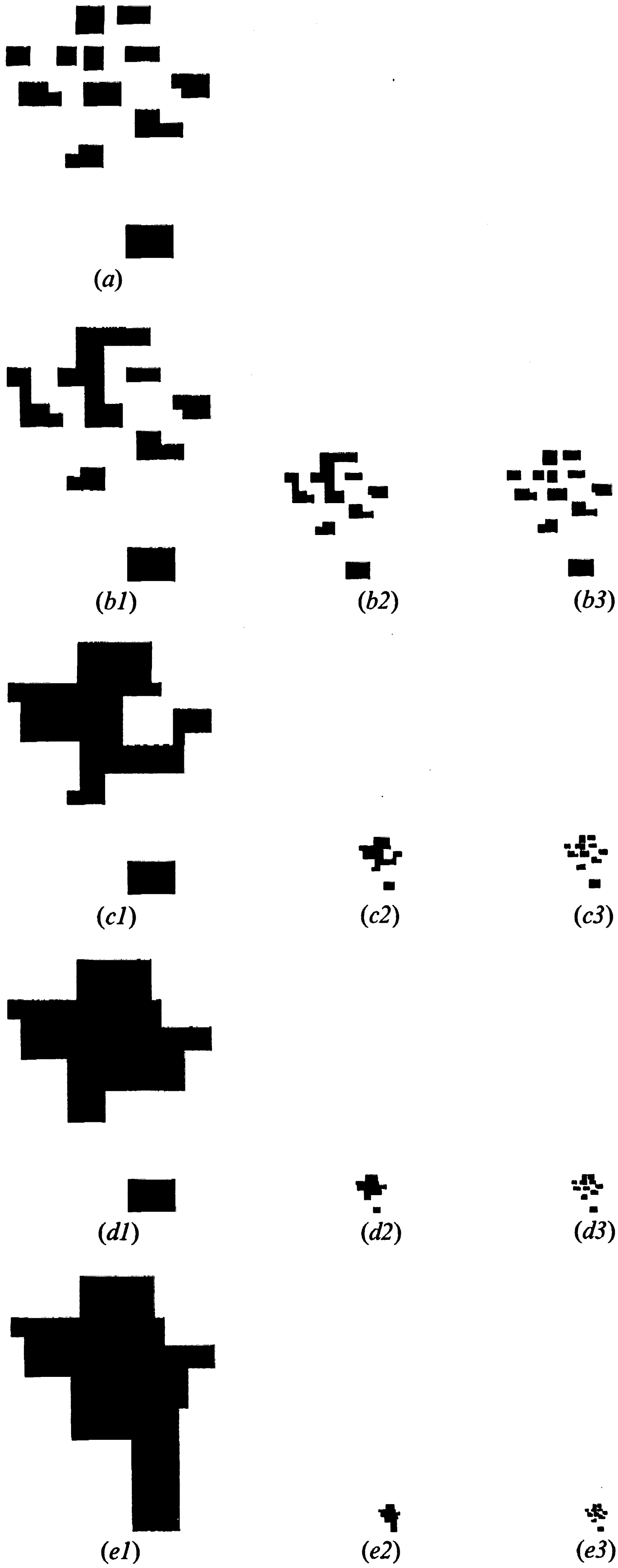
5. Algebraic models for shape refinement

In the previous section, the first step of the aggregation process, i.e., area combination, was discussed. The algebraic model is given in equation (8) and the determination of the size and shape of the structuring elements for this model have been

(*a*) (*b*) (*c*) (*d*)

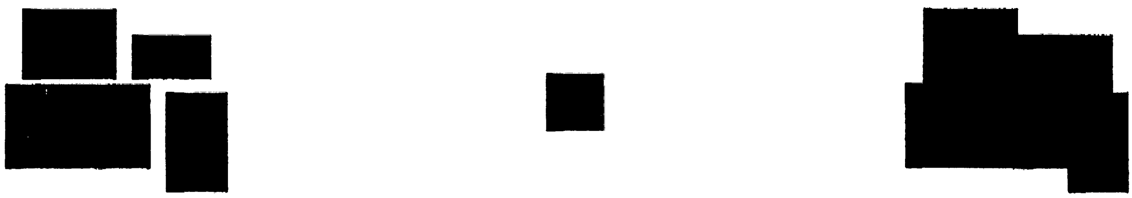
Figure 9. Structuring elements used at different scales. (*a*) For 2 × reduction, (*b*) For 5 × reduction, (*c*) For 7 × reduction, (*d*) For 10 × reduction.

Figure 10. Combination of area features at different scales. (*a*) Area features to be aggregated, (*b*1) Combined for 2 × scale reduction, (*b*2) Combined image 2 × reduced, (*b*3) Original image 2 × reduced, (*c*1) Combined for 5 × scale reduction. (*c*2) Combined image 5 × reduced, (*c*3) Original image 5 × reduced, (*d*1) Combined for 7 × scale reduction, (*d*2) Combined image 7 × reduced, (*d*3) Original image 7 × reduced, (*e*1) Combined for 10 × scale reduction, (*e*2) Combined image 10 × reduced, (*e*3) Original image 10 × reduced.


(a)
(b1)
(b2)
(b3)
(c1)
(c2)
(c3)
(d1)
(d2)
(d3)
(e1)
(e2)
(e3)

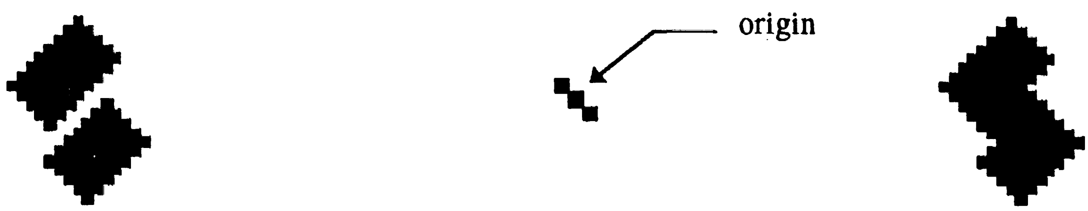
(a) (b) (c)

Figure 11. Combination of rectangular features (I). (a) Original features A, (b) Structuring element $B = B_1 = B_2$, (c) Combined area $C = A \cdot B$.



(a) (b) (c)

Figure 12. Combination of rectangular features (II). (a) Original features A, (b) Structuring element $B = B_1 = B_2$, (c) Combined area $C = A \cdot B$.



(a) (b) (c)

Figure 13. Combination of inclined rectangular features. (a) Original features A, (b) Structuring element $B = B_1 = B_2$, (c) Combined area $C = A \cdot B$.

(a) (b) (c)

Figure 14. Combination of curved area features (I). (a) Original features A, (b) Structuring element $B = B_1 = B_2$, (c) Combined area $C = A \cdot B$.

discussed in §4.3 and 4.4. This section will discuss the second step, i.e. shape refinement.

It can be seen from figure 10 to figure 17 that, in some cases, the area features resulting from the combination process may appear very irregular, so that a shape refinement may need to follow to satisfy the graphic presentation. This is especially the case if the difference between source scale and target scale is large and where many small features are combined. Figure 10(e2) is an example. There are two solutions for shape refinement. The first is to apply a sequential thickening operator to obtain the convex hull of the combined object, and the second is to apply the opening operation with a combination of other operators to simplify the shape. These two methods will be discussed in the next two sub-sections.

Figure 15. Combination of curved area features (II). (*a*) Original features A, (*b*) Structuring elements, (*c*) Areas combined $C = A \cdot B_1$, (*d*) Combined area $C = (A \oplus B_1) \ominus B_2$.

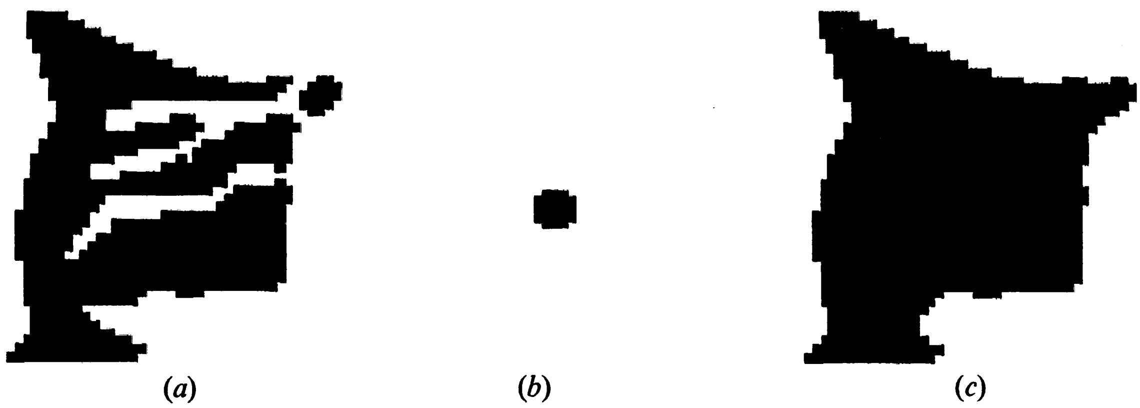


Figure 16. Combination of curved feature area features (III). (*a*) Original features A, (*b*) Structuring element B, (*c*) Combined area $C = A \cdot B$.

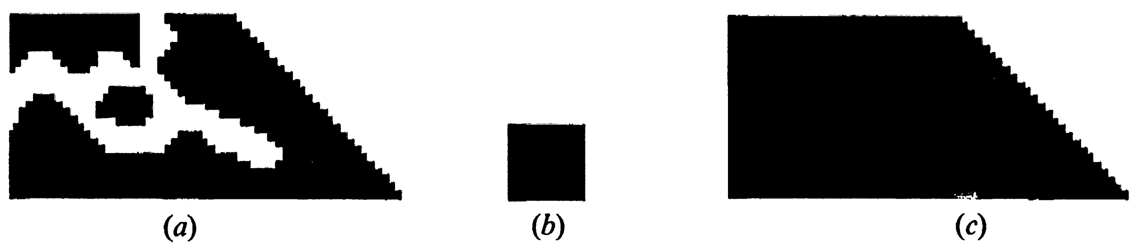


Figure 17. Combination of curved area features (IV). (*a*) Original feature A, (*b*) Structuring element B, (*c*) Combined area $C = A \cdot B$.

5.1. *Convex hull formation*

The algebraic model for forming the convex hull of an irregular area feature is as follows:

$$C_h = C \circledcirc \{B_i\} \tag{11}$$

where $\circledcirc$ is an operator called thickening, through which the original image will be thickened. $\{B_i\}$ means a series of special structuring elements for thickening purposes, which are shown in figure 18. The thickening operator is a combination of a number

Figure 18. The series of structuring elements for convex hull formation ("x" means "don't care").

Figure 19. Shape refinement of an area feature—the convex hull. (*a*) The area feature combined and reduced as shown in figure 10(e2), (*b*) The area feature refined by a convex hull.

of other morphological operators. This requires in-depth discussion beyond the scope of this paper, for which further information can be found in Serra (1982).

This model (i.e., equation (11)) means that the convex hull is formed through sequential thickening by a set of structuring elements $\{B_i\}$. By applying equation (11) to the area feature resulting from the combination process shown in figure 10(*e*2), the result as shown in figure 19(*b*) can be obtained.

5.2. *Irregularity filtering*

The above solution is suitable for those shapes which are very close to rectangular, or where a rectangular shape is required for the final result (e.g., in the case of a block of buildings). However, it doesn't necessarily work well for other cases. Indeed, in other cases, the following procedure, consisting of a set of morphological operators, will produce more realistic results:

(*a*) To eliminate those small convex areas on feature C, using an opening operator:

$$D = C \circ B \tag{12}$$

(*b*) To form the convex hull of the opened area D:

$$C_h = D \circledcirc \{B_i\} \tag{13}$$

(*c*) To obtain the complementary set of the opened area within the convex area:

$$E = C_h - D \tag{14}$$

(*d*) To eliminate small convex areas on the area E, using an opening operator:

$$F = E \circ B \tag{15}$$

(*e*) To obtain the complementary set of F:

$$G = C_h - F \tag{16}$$

where B is a structuring element and is different from $\{B_i\}$. Here, the size of B should be only half the size of the structuring elements used in figure 9.

This is a roundabout procedure. The ultimate aim is to cut off small convex

Figure 20. Shape refinement for an area feature—an alternative. (*a*) Combined and reduced image as shown in figure 10 (e2), (*b*) Refined area feature.

areas and fill up small concave areas. By applying this procedure to the resulting feature shown in figure 10(*e*2), a result as shown in figure 20(*b*) is obtained. This result looks more reasonable.

6. Conclusions

In this paper, one of the many operations for area features, i.e. aggregation, is discussed. The aggregation operation is decomposed into two components, viz., a combination sub-process and a shape refinement sub-process. Algebraic models for this generalization operation are built upon the operators developed in mathematical morphology.

The algebraic model for the first sub-process involves combining area features according to scale. The key to success is the correct determination of the shape and size of the structuring elements to be used. The size of the structuring element is dependent on source scale and target scale of the spatial data. Regarding the shape of the structuring element, it is recommended that circular shape should be used for curved area features and rectangular shape for rectangular area features.

For the second sub-process, i.e., to smooth out irregularities, two solutions are outlined. One is to form a convex hull of the area feature resulting from the combination operation, and the other is to eliminate small variations along the boundary using a more complex procedure. The convex hull method is only suitable for the cases when the final result should be a rectangular shape. These algebraic models seem to work well, as revealed in examples.

However, no claim is made by the authors that the results obtained from these algebraic models are clearly superior to results from other methods. Rather, the main aim of this paper is to offer some techniques and an alternative for area aggregation which is mathematically more elegant than other conventional methods, so that a mathematical basis might be established for this generalization operation. Indeed, in addition to their mathematical elegance, these models are also more generic than those developed using expansion and shrink operators, because as demonstrated these two operators are only special cases of dilation and erosion.

Finally, it should also be noted that this paper deals only with geometric issues and the algebraic models operate at a very basic level. To make a generalization system successful, higher level semantic information and other cartographic knowledge is needed to control the effect of the low-level models described in this paper. Indeed, this should be a topic for future research.

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