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MATHEMATICAL MODELS OF THE ACCURACY OF DIGITAL TERRAIN MODEL SURFACES LINEARLY CONSTRUCTED FROM SQUARE GRIDDED DATA

By ZHILIN LI
Technische Universität Berlin

Abstract

This paper presents two mathematical models of the accuracy of digital terrain model (DTM) surfaces which have been linearly constructed from square gridded data, respectively with and without additional feature specific data. In these models, the accuracy of DTM surfaces is expressed as a function of a few parameters, such as slope angle, grid interval or the accuracy of raw data, which are familiar to practitioners in photogrammetry and other mapping related disciplines. These models are in a form similar to the conventional formulae for contour map accuracy and they are compared with experimental test results.

INTRODUCTION

IN THE PRACTICE of digital terrain modelling, the accuracy of the resulting digital terrain model (DTM) surfaces is possibly the most important concern to both DTM producers and users. Such an accuracy may be either predicted by a mathematical model or checked experimentally against a corresponding set of control points which are used as ground truth. A mathematical model of the accuracy of a DTM surface is beneficial from both the theoretical and practical points of view, if it is capable of producing reliable prediction. Moreover, a DTM project can be completed in an economic and efficient manner only if such a model is available. Unfortunately, existing DTM accuracy models have serious limitations with regard to prediction (Balce, 1987; Li, 1990, 1993), although many models have been developed, for example by Makarovič (1972), Kubik and Botman (1976), Frederiksen (1980), Tempfli (1980) and Frederiksen *et al.* (1986). In addition, Li (1990, 1993) has also pointed out that the parameters used in these models are either difficult to estimate or are unfamiliar to the mapping community. Therefore, the applicability of these models is very limited; this study attempts to develop a family of new models which is capable of producing reliable results in practice.

In this paper, firstly a brief discussion is given of the main factors which affect the accuracy of DTM surfaces. Error propagation in the modelling process is described and a general mathematical model is formulated of the accuracy of DTM surfaces linearly constructed from grid data, both with and without additional feature specific data. A discussion is presented concerning how the parameters in the general model can be estimated and finally, predicted accuracies from the mathematical models are compared with experimental results.

THE MAIN FACTORS AFFECTING DTM ACCURACY

Errors in DTM data points are the accumulated result of errors introduced in all the operations of the digital terrain modelling process which are propagated from various sources. Li (1990, 1992) has identified the importance of the following factors:

(1) the characteristics of the terrain surface;
(2) the methods used for constructing the DTM surface;
(3) the three attributes (accuracy, distribution and density) of the source (or raw) data; and
(4) the characteristics of the resulting DTM surface.

The characteristics of the terrain surface define the degree of difficulty in representing the terrain surface and these features therefore play a significant role in the accuracy of the resulting DTM surfaces. Slope has been found to be the most important descriptor of terrain surface and it is widely used in surveying and mapping practice. Thus, in this study, slope and wavelength (horizontal variation) are used in combination to describe the surface. The acceptability of using such parameters has been justified by Li (1993).

A DTM surface can be constructed by two methods: either directly from the measured data (measured grid data) or indirectly from the grid data (for example a square grid or triangular grid) derived as a result of so called random-to-grid interpolation at a preprocessing stage. In this study, only measured grid data are considered.

There is no doubt that errors in source data (grid nodes in gridded data) will be propagated through the modelling process to the resulting DTM surfaces. Such errors can be described in terms of variance (σ^2_{nod}) and covariance. If the measurement of each point is considered as being independent, the covariance can be neglected. Indeed, covariance between photogrammetrically measured data is difficult to estimate and is usually neglected in practice; it is also ignored in this study since only measured grid data are considered.

The distribution of the source data is another major factor affecting the accuracy of the resulting DTM surface. This parameter can be described by the pattern, location and orientation of the data set. In this study, only one particular pattern, the square grid, is of concern since it is still the most popular grid set. Furthermore, feature specific data points (such as peaks, pits, points along ridge lines and ravine lines, passes, points along break lines, and so on) may be added to square gridded data, thus forming so called composite data. Composite data formed in this manner will also be considered in this study, but orientation and location are not taken into account.

The density of the source data is perhaps the most important factor (Kubik and Botman, 1976; Ackermann, 1980; Li, 1992). It can be specified by such parameters as average points interval, number of points per unit area and the cut frequency (Nyquist frequency) of the spatial variation represented by the data set. In the case of the square grid, the grid interval (denoted as d) is an appropriate measure. Even in the case of composite data, this parameter is still representative and it is therefore used in this study.

The characteristics of the resulting DTM surface represent a factor determining the "goodness of fit" of the DTM surface to the terrain surface, thus defining the accuracy of the DTM surface. It should be noted that DTM surfaces can be either continuous or discontinuous, and either smooth (using a higher order polynomial) or non-smooth (using sets of linear surfaces). It has been recognized by many researchers (for example Peucker (1972)) that linear surfaces are usually the least misleading; these are continuous surfaces comprising either contiguous bilinear surfaces or triangular facets or a hybrid of both. This type of surface is selected as a typical example for the present investigation.

Summarising, in this study the accuracy of a DTM surface will be modelled by considering (1) the propagation of errors from measured square grid data through linear modelling using the direct construction method and (2) the accuracy loss due to the linear representation of the terrain surface.

Error Propagation in Linear Modelling

The linear modelling of square grids means that contiguous bilinear facets are constructed to represent the terrain surface. The height value of a point at a desired position is then interpolated from a bilinear surface.

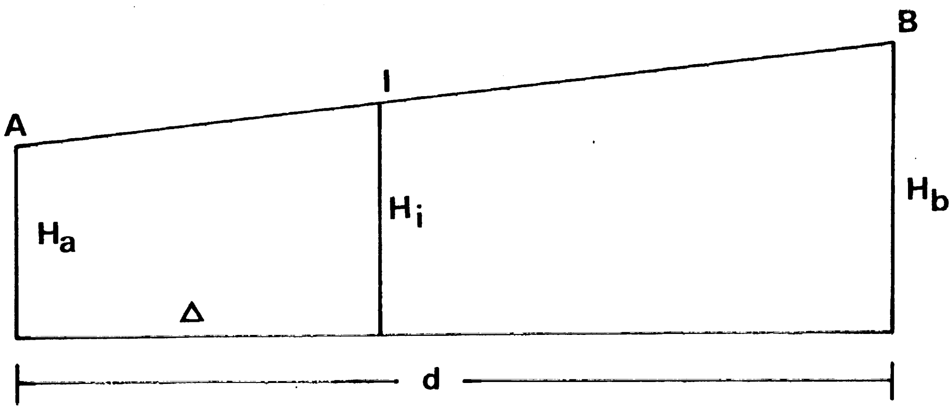

FIG. 1. Linear interpolation of point I between points A and B.

Error Propagation Along a Profile

When discussing error propagation in linear modelling, error propagation in a profile should first be considered. Suppose A and B in Fig. 1 are two grid nodes with an interval of d and point I, between points A and B, is to be interpolated. If the horizontal distance from I to Point A is Δ, then

$$H_i = \frac{d-\Delta}{d} H_a + \frac{\Delta}{d} H_b \tag{1}$$

where H_a and H_b are the heights of points A and B respectively, and H_i is the interpolated height of point I. If points A and B are measured with an accuracy of σ^2_{nod} in terms of error variance, then the variance of the errors for point I, σ_i^2, which are propagated purely from two grid nodes, can be expressed as follows:

$$\sigma_i^2 = \left(\frac{d-\Delta}{d}\right)^2 \sigma^2_{nod} + \left(\frac{\Delta}{d}\right)^2 \sigma^2_{nod}. \tag{2}$$

Equation (2) is an expression for the accuracy (in terms of error variance) of a particular point with a specific location along a side of a bilinear surface. However, what is of interest here is the overall average value for all possible points along the line AB, which is a representative value for the DTM profile. In this case, the horizontal distances of these points to point A in Fig. 1, (Δ in equation (2)) should be considered as a variable which takes a value from 0 (at point A) to d (at point B). Therefore, the average value for the variances of all the points between A and B can be computed as follows:

$$\sigma_S^2 = \frac{1}{d}\int_0^d \left(\left(\frac{d-\Delta}{d}\right)^2 \sigma^2_{nod} + \left(\frac{\Delta}{d}\right)^2 \sigma^2_{nod}\right) d\Delta \tag{3}$$

$$= \frac{2}{3}\sigma^2_{nod}$$

where σ_s^2 is the overall average value of error variances for all the points along the whole profile with a grid interval of d, but only with respect to errors propagated from source data, in other words grid nodes.

For the overall accuracy of the points along a profile, another term concerning accuracy loss due to the linear representation of the terrain surface should be added, thus giving the following formula:

$$\sigma^2_{Pr} = \sigma_S^2 + \sigma_T^2 = \frac{2}{3}\sigma^2_{nod} + \sigma_T^2 \tag{4}$$

where σ_T^2 denotes the accuracy loss due to the linear representation of terrain surface in terms of variance (which will be discussed later), σ^2_{nod} is the variance of the errors

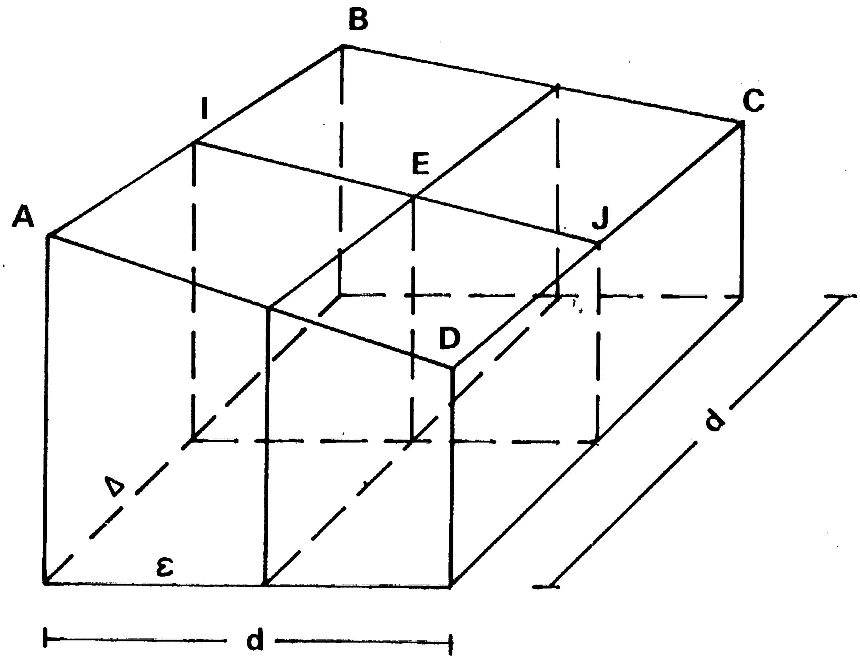

FIG. 2. Bilinear interpolation of point E using four nodes (A, B, C and D).

at grid nodes and σ^2_{Pr} denotes the overall accuracy of the DTM points along the profile with an interval of d in terms of error variance.

Error Propagation on a Bilinear Surface

In the case of a bilinear surface, the interpolation of a point takes place in two perpendicular directions. Suppose the four nodes are points A, B, C and D (Fig. 2) and point E is to be interpolated on the bilinear surface. The interpolation could take place initially along AB and DC, using equation (1). Thus, point I can be interpolated from A and B and similarly point J can be interpolated from D and C. The next step of the interpolation takes place between points I and J as follows:

$$H_e = \frac{d-\varepsilon}{d} H_i + \frac{\varepsilon}{d} H_j \tag{5}$$

where ε is the horizontal distance from point E to point I, and H_e, H_i and H_j are the height values of points E, I and J respectively.

Thus equation (5) again expresses linear interpolation along a profile with an interval of d. Fundamentally, it is identical to equation (1). Therefore, the same development as for equation (1) can be made and a formula similar to equation (3) can also be obtained. However, the accuracy of points I and J in Fig. 2, as for point I in Fig. 1, is different from that of points A, B, C and D; the actual accuracy value varies with the positions of I and J between the two node points and the characteristics of the terrain surface. Therefore, the average value expressed by equation (3), σ^2_{Pr}, should be used as the representative for points I and J in Fig. 2. Again, there is an accuracy loss (σ^2_T) due to the linear representation for profile IJ. Thus, an analogue to equation (4) can be obtained for the accuracy of the points interpolated from a bilinear surface as follows:

$$\sigma^2_{Surf} = \frac{2}{3}\sigma^2_{Pr} + \sigma^2_T. \tag{6}$$

By substituting equation (4) into equation (6), the following expression can be obtained:

$$\sigma_{Surf}^2 = \frac{2}{3}\left(\frac{2}{3}\sigma_{nod}^2 + \sigma_T^2\right) + \sigma_T^2$$
$$= \frac{4}{9}\sigma_{nod}^2 + \frac{5}{3}\sigma_T^2 \quad (7)$$

where σ_{Surf}^2 denotes the average value for the accuracy of the points on a bilinear surface, σ_{nod}^2 is the accuracy of the node and σ_T^2 denotes the accuracy loss due to the linear representation of terrain profiles, all in terms of error variance.

By comparing equation (4) with equation (7), it can be shown that the coefficient for σ_{nod}^2 in equation (7) is even smaller than that in equation (4). This is because, in bilinear interpolation, more node points have been used than for the profile case (four points compared with two points). As an example, consider the case where the middle point on a bilinear surface is interpolated; the average height of the four nodes is the computed result and the accuracy of this interpolated point is only $(1/4)\sigma_{nod}^2$. However, for the middle point of a profile, the interpolated height is the average value of the two nodes and thus its accuracy is $(1/2)\sigma_{nod}^2$. The former accuracy is twice as small as the latter.

Accuracy Loss due to Linear Representation of Terrain Surface

So far, the general form of the accuracy model has been derived and expressed by equation (7). In this connexion, two important problems need to be solved: (a) the accuracy of grid nodes (σ_{nod}^2) and (b) the accuracy loss due to linear representation of terrain surface (σ_T^2). It is not difficult to estimate a value for σ_{nod}^2. For example, for photogrammetric data measured in static mode, the approximate value for accuracy is 0·07 *H*‰ to 0·1 *H*‰ (per mille of flying height) for an analytical plotter and 0·1 *H*‰ to 0·2 *H*‰ for a precision analogue plotter. For data measured in dynamic mode, the expected result would be 0·3 *H*‰. Therefore, the only remaining problem is to obtain a good estimate of σ_T^2.

Strategy for Determination of σ_T^2

Terrain shape obviously varies from place to place and it is therefore impossible to depict its inflexions using an analytical method, especially for small local deviations. These characteristics can only be handled by using statistical methods.

In the case of the linear modelling of a terrain surface, σ_T should represent the standard deviation of all the height differences (δh) between the terrain surface and the resulting linear facets (the DTM surface) which is constructed from error free nodes. In this case, δh is a random variable. According to statistical theory, for a random variable (such as δh), regardless of its distribution, its σ value (σ_T here) gives a strong indication of its dispersion. Expressed mathematically:

$$P(|\delta h - \mu| \leq K\sigma_T) \geq f(K) \quad (8)$$

where μ is the mean value, K is a constant and $f(K)$ is a function of K with its value ranging from 0 to 1. Suppose δh has a normal distribution; if K takes a value of 3, then $f(K)$ is equal to 99·73 per cent. This means that for normal distribution, with a probability of 99·73 per cent, δh will have a value (if sampled) from $-3\sigma + \mu$ to $3\sigma + \mu$. This probability is so large that, in error theory, 3σ is regarded as the maximum error and any error greater than this value is considered as a gross error. Taking an analogue to the practice of error theory, the following expression seems appropriate:

$$\sigma_T = \frac{E_{max}}{K} \quad (9)$$

where σ_T is the accuracy loss due to linear representation of terrain profiles, E_{max} is the possible maximum error (which will be discussed later) and K is the same constant as given in equation (8). The value of K depends on the distribution of δh. In the above example of normal distribution, a value of 3 is considered as an appropriate value.

The next problems are: (1) to estimate E_{max}; and (2) to obtain an appropriate value for K.

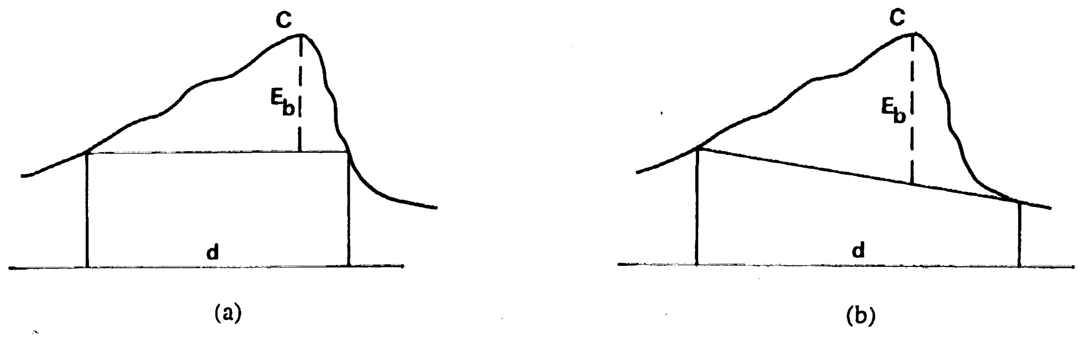

FIG. 3. Possible maximum errors of linear representation, due to terrain faults or breaks, with different locations of grids.

Extreme Errors (E_{max}) due to Linear Representation

In order to analyze the possible values for extreme values of δh, some possible outlines of terrain profiles in extreme situations need to be considered. Since only extreme cases are being examined, some of the analysis may seem unrealistic.

Figs. 3(a) and 3(b) illustrate the maximum possible errors at point *C*, for the same terrain feature but with different locations of nodes, due to a fault or other geological structure giving rise to a steep change in slope. If information giving a full description of this structure has not been collected, a huge error may result. The value of such an error, denoted as E_b here, varies with the characteristics of the terrain feature itself. These values can only be measured directly and not estimated analytically.

Figs. 4(a) and 4(b) show the possible positive maximum error at point *C* for different locations of nodes when only points which are located on regular grid nodes are sampled (in other words without feature specific points). As shown in Fig. 4(a), the possible maximum value of E_r arises when point *C* lies in the middle of the grid and it can be computed as follows:

$$E_{r,\,max} = CB = \frac{1}{2}\, d \tan\beta. \qquad (10)$$

$E_{r,\,max}$ represents the possible maximum error in such a case. Similarly, the possible negative error can also be estimated.

Fig. 5(a) shows the possible errors which may occur for grid data with some feature specific data for a convex slope. This figure can be justified because it is not practical to include all convex and concave points, even for the case where pure selective sampling has been carried out on a stereomodel (in a photogrammetric

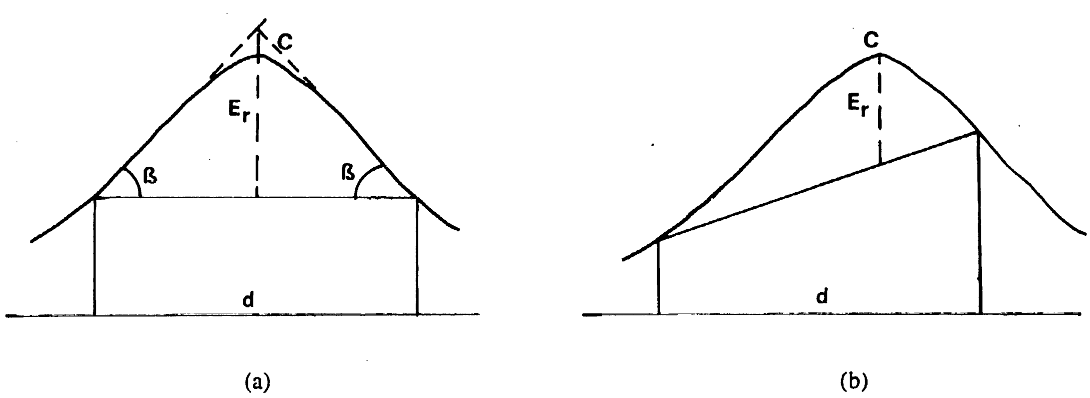

FIG. 4. Possible maximum errors of linear representation using only grid nodes, with different locations of grids: (a) shows that the maximum value occurs when a grid contains local maxima or minima; (b) shows that E_r varies with the location of grids.

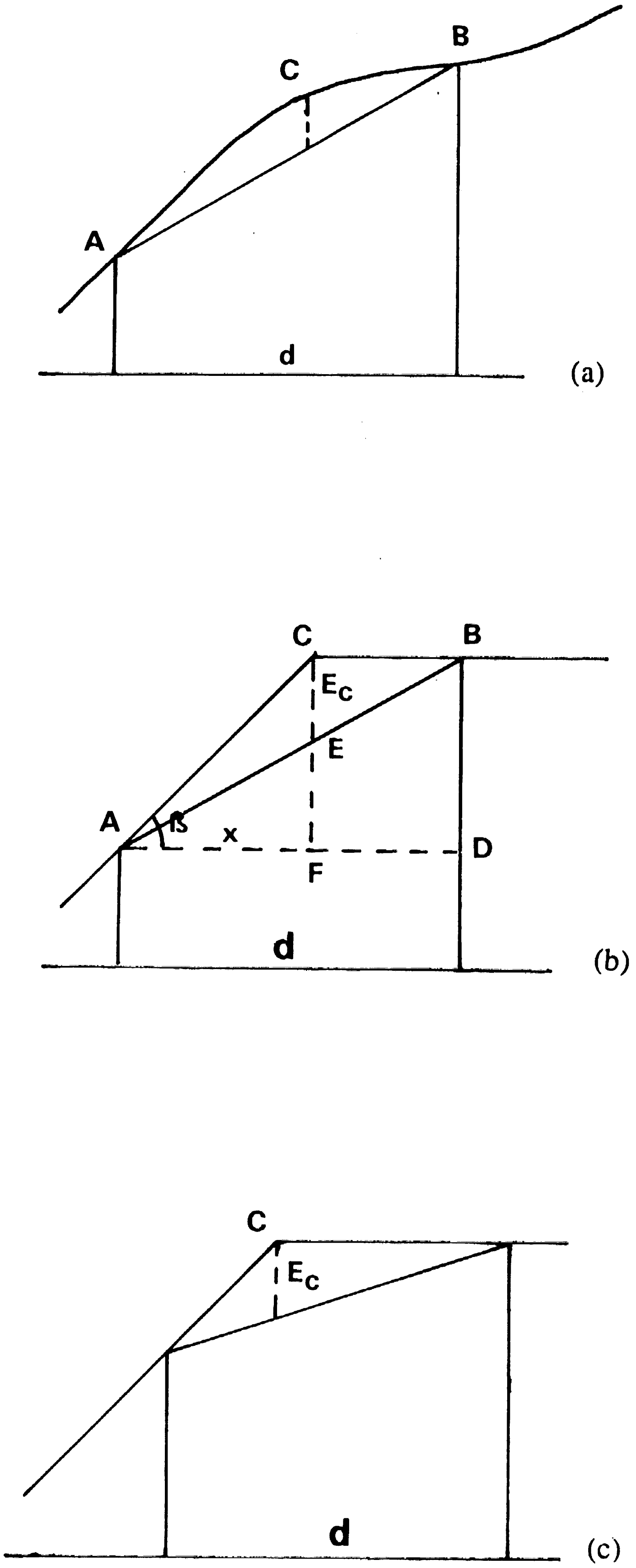

FIG. 5. Possible maximum errors of linear representations of ordinary terrain slope. (a) is a convex slope; (b) is exaggerated from (a) for the convenience of analysis; (c) shows that E_c varies with location of grids.

instrument) to simulate data acquisition by ground survey. Fig. 5(b) is exaggerated from Fig. 5(a) for the convenience of obtaining a numerical estimate. Point C in this diagram shows an extreme case of convex slope. Line AB is the linearly constructed profile; angle CAD is the slope angle at point A (denoted as β) and line segment CE is the possible error at point C. Therefore,

$$CE = CF - EF = X \tan\beta - \frac{X^2 \tan\beta}{d}. \tag{11}$$

Fig. 5(c) also shows that the value of E_c varies with the location of grid nodes. The next task is to find the maximum value for CE (see Fig. 5(b)) representing all different possible locations of point C in terms of horizontal distance from point A. If the first derivative of CE is considered to be equal to zero, then the location of C where the value of CE reaches its maximum can be determined as follows:

$$\frac{d(CE)}{dX} = \tan\beta - \frac{2X \tan\beta}{d} = 0. \tag{12}$$

From equation (12) it can be seen that $X = d/2$. By substituting this value into equation (11) and denoting CE as E_c, then

$$E_{c,max} = CB = \frac{1}{4} d \tan\beta. \tag{13}$$

Therefore, it can be deduced that the value of possible extreme errors for the case of regular grid data only is double that for the case of composite data. The maximum error due to linear representation is $E_{c,max}$ for composite data, whereas for grid data only, the situation is more complicated.

A Practical Consideration Regarding E_{max} and σ

The three extreme errors identified in the previous section belong to three different distributions. E_b applies to grids taken across faults and break lines; E_r relates to grids taken across peaks, pits, ridges and ravines; and E_c is used for ordinary terrain features and therefore for all the remaining grids. Suppose the proportions of grids which may contain E_c, E_r and E_b are $P(c)$, $P(r)$ and $P(b)$, then

$$P(c) + P(r) + P(b) = 1. \tag{14}$$

For composite data, $P(r)$ and $P(b)$ are zero. It is only necessary to estimate $P(r)$ and $P(b)$ for regular grids. If no faults such as Fig. 3 occur, then $P(b)$ is zero. Otherwise, $P(b)$ can be estimated according to the height over the length and width of the faults or breaks.

Similarly, the estimation of $P(r)$ is not an easy task. For a small area, there is no better method than simply counting the number of grid cells across the ridge and ravine lines and then dividing by the number of total grids. For a large area, some alternatives may be used. The value of $P(r)$ is directly related to the wavelength of terrain variation (Fig. 6). However, the planimetric shape of a hill (expressed by its contours) could be very different from place to place. Even for the same hill, the wavelength could be different if the profiles are taken along different directions. Therefore, even a rough estimate, such as an average value, could be valuable. The value of the average wavelength can be estimated as follows:

$$\lambda = 2H \cot\alpha \tag{15}$$

where H is the average relative height, α is the average slope angle and λ is the average wavelength, all taken over the entire area to be modelled (Fig. 6). In practice, the average value of local relief (half of the maximum minus the minimum heights) can be used to represent H, such that

$$\lambda = (H_{max} - H_{min}) \cot\alpha. \tag{16}$$

Once the estimation of λ has been made, the value of $P(r)$ can be estimated. Both the top and the bottom of a spatial variation will occur over a single wavelength in one profile direction. Therefore, for a grid which has two profile directions perpendicular to each other, the occurrence frequency of the E_r values over a grid is as follows:

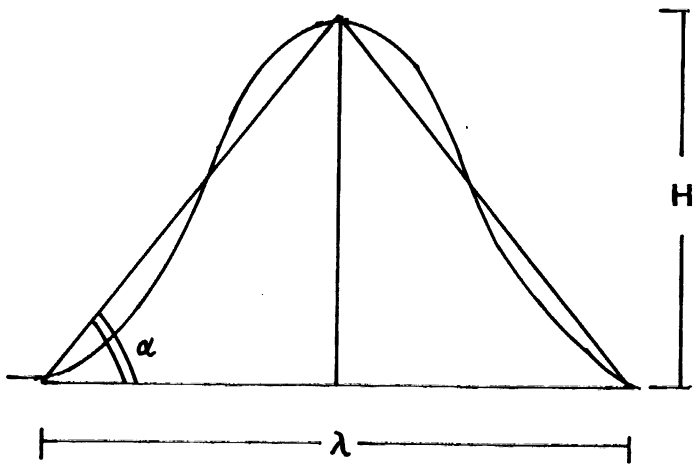

FIG. 6. Approximate estimation of wavelength, λ, where H is the average of the height variations.

$$P(r) = \frac{4d}{\lambda} \tag{17a}$$

where λ is the average wavelength, d is the grid interval and $P(r)$ is the occurrence frequency of E_r. An idealized diagram such as Fig. 7 may help in understanding the estimation of $P(r)$. In this example, the total number of grid squares is $(1{\cdot}5\lambda/d) \times (1{\cdot}5\lambda/d)$. Suppose all the profiles along both directions are identical to the ones shown in Fig. 7, then the total number of grid squares which may contain E_r is, as marked in Fig. 7, approximately equal to $6(1{\cdot}5\lambda/d)$. Thus $P(r)$ is $4d/\lambda$, which is expressed by

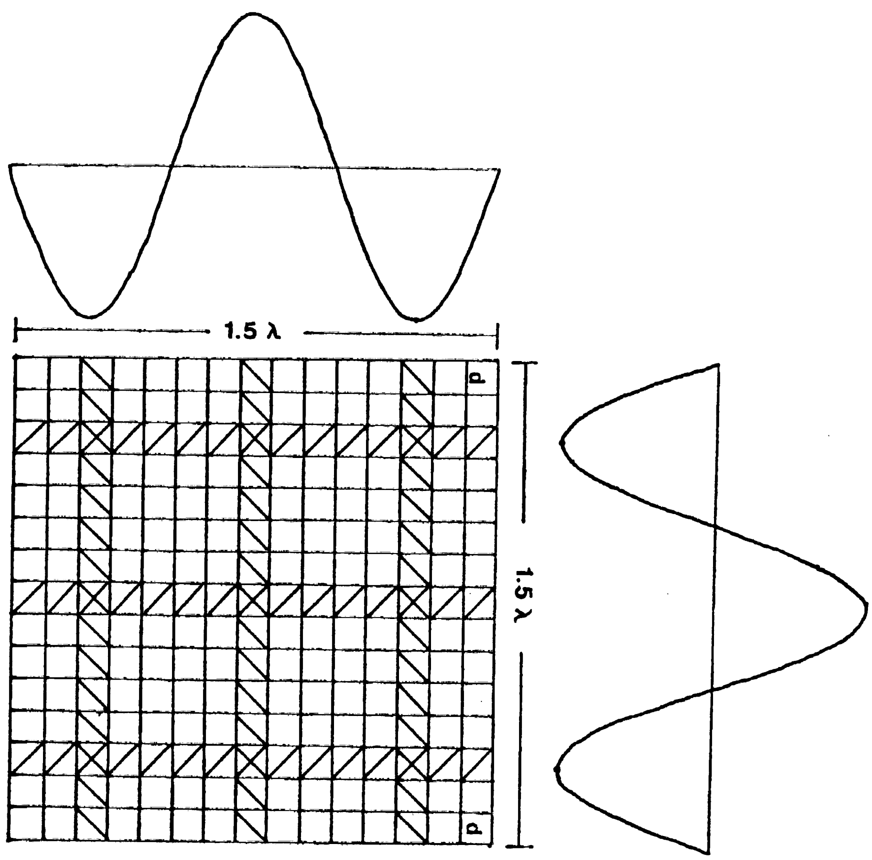

FIG. 7. Approximate estimation of $P(r)$, which represents the proportions of grids containing local maxima and minima.

equation (17a). However, a more important consideration is for the area with size $\lambda \times \lambda$. Fig. 7 shows that, in this unit area, $P(r) = 4d/\lambda = d(4\lambda/\lambda^2)$. Here 4λ is the perimeter and λ^2 is the area. This leads the author to suspect that the following expression for $P(r)$ is more general and more appropriate:

$$P(r) = d\frac{\textit{perimeter of lowest contour}}{\textit{area enclosed by lowest contour}}. \tag{17b}$$

Equation (17b) could be very useful for the estimation of $P(r)$ from existing contours which from the map appear to be of any irregular shape.

Therefore, for the DTM linearly constructed from grid data only, the value of σ_T can be estimated as follows:

$$\sigma_T = \frac{P(r)\,E_{r,\,max} + P(c)\,E_{c,\,max} + P(b)\,E_{b,\,max}}{K}. \tag{18}$$

It may be argued that such an averaging operation is not justified from a statistical point of view, since E_b, E_r and E_c belong to three different distributions. However, in DTM practice, it is never possible to distinguish between these three types of errors and estimates are always made from a sample which contains all of them. Therefore, equation (18) is an appropriate representation of DTM practice.

In practical terms, E_b rarely occurs and even if it does occur, it is normally sampled. Thus, it is acceptable to neglect E_b in equation (18), giving:

$$\begin{aligned} \sigma_T &= \frac{P(r)\,E_{r,\,max} + P(c)\,E_{c,\,max}}{K} \\ &= \frac{P(r)\,E_{r,\,max} + (1 - P(r))\,E_{c,\,max}}{K}. \end{aligned} \tag{19}$$

Finally it is necessary to determine an appropriate value for K.

Estimating the Value of K

If the distribution of the errors due to linear representation was known, then it would be easy to obtain a good estimate for K. However, the problem is that such a distribution of errors has still not been established. In some cases, it seems to follow a normal distribution while in other cases this is not true (Torlegård *et al.*, 1986; Li, 1992). The usual assumption of normal distribution in error theory is not necessarily applicable and therefore a value of 3 is not necessarily an appropriate value for K.

From a theoretical point of view, according to Chebyshev's law, the majority of errors are massed with a range from $-4\sigma + \mu$ to $4\sigma + \mu$. The probability is at least 94 per cent regardless of distribution. In the case of terrain modelling, which is close to a normal distribution, such a probability is therefore much greater. Thus a value of four could be taken as an approximate value for K.

Results obtained from experimental tests on the accuracy of DTM surfaces carried out by the author indicate that a value of four could be the appropriate value for K in these circumstances. The occurrence frequency for DTM errors greater than 4σ is from 0·25 per cent to 0·30 per cent. These results were obtained from 74 tests (using 74 data sets), each with more than 1500 errors (taken as residuals at check points), which were obtained from three different test areas. A residual error is, of course, slightly different from σ since the former is also affected by both the errors on grid nodes and the errors at check points. However, in these tests, the check points were observed with much greater accuracy so that their influence in comparison with σ_T is very small. Thus these results must represent a true picture. Although such a sample cannot be considered as being very large, and the examples are not very comprehensive, the investigation gives some insight into the error distribution. Therefore, from both a theoretical and a practical point of view, a value of 4 seems valid for K in equation (8).

Mathematical Expressions of DTM Accuracy Models

It has been shown that accuracy loss due to linear representation for a digital terrain model linearly constructed from measured grid data only can be written as follows:

$$\begin{aligned}\sigma_{T,r} &= \frac{E_{c,max}}{K}(1-P(r)) + \frac{E_{r,max}}{K}P(r) \\ &= \frac{d\tan\alpha}{4K}\Big(1-P(r)\Big) + \frac{d\tan\alpha}{2K}P(r) \\ &= \frac{d\tan\alpha}{4K}\Big(1+P(r)\Big) \\ &= \frac{d\tan\alpha}{4K}\left(1+\frac{4d}{\lambda}\right).\end{aligned} \tag{20}$$

Moreover for a DTM constructed from composite data, the accuracy loss formula is as follows:

$$\sigma_{T,c} = \frac{E_{c,max}}{K} = \frac{d\tan\alpha}{4K}. \tag{21}$$

Substituting equation (21) and equation (20) into equation (7), the accuracy of digital terrain models linearly constructed respectively from composite data and grid data only and is as follows:

$$\sigma^2_{Surf/c} = \frac{4}{9}\sigma^2_{nod} + \frac{5}{48K^2}(d\tan\alpha)^2 \tag{22a}$$

$$\sigma^2_{Surf/r} = \frac{4}{9}\sigma^2_{nod} + \frac{5}{48K^2}\Big(1+P(r)\Big)^2(d\tan\alpha)^2 \tag{22b}$$

where $\sigma^2_{Surf/c}$ and $\sigma^2_{Surf/r}$ denote the accuracies of digital terrain models linearly constructed from composite data and from grid data only (both in terms of error variance), σ^2_{nod} is the variance of errors at measured grid nodes, K is a constant (approximately equal to 4 depending on the characteristics of the terrain surface), α is the average slope angle of the area and $P(r)$ is the proportion of grids which may contain E_r (which is expressed by equation (17)).

At this point the derivations of the formulae have all been completed. However the discussion can be extended to provide an approximation of equation (22) as follows:

$$\sigma_{Surf/c} = \frac{2}{3}\sigma_{nod} + \frac{\sqrt{5}}{\sqrt{48}K}(d\tan\alpha) \tag{23a}$$

$$\sigma_{Surf/r} = \frac{2}{3}\sigma_{nod} + \frac{\sqrt{5}}{\sqrt{48}K}\Big(1+P(r)\Big)(d\tan\alpha). \tag{23b}$$

These equations represent an analogue to the Koppe formulae which are widely used for specifying map accuracy in middle European countries. Equations (23) prove to be a very good approximation of equations (22) in the case where grid intervals are relatively small and equations (23) are more convenient to use in practice.

Evaluations of Accuracy Models

Once the mathematical models of the accuracy of DTM surfaces have been established, their effectiveness needs to be judged. In this study, three sets of experimental results are used for this purpose. Details of how the tests were designed and carried out, the results that have been obtained and an analysis (both descriptive and regression) of these results have been presented in another paper (Li, 1992). It is important to note here that the test results were obtained using a triangulation based package. In other words, the constructed DTM surface is comprised of triangular facets, not contiguous bilinear surfaces. However the mathematical models described in this paper are for bilinear surfaces and it may therefore be argued that such an evaluation is not comprehensive. Nonetheless, the results definitely offer a suggestion of the aptness of these mathematical models.

The test areas selected were Uppland (Sweden), Sohnstetten (Germany) and Spitze (Germany), which were those used for DTM tests carried out by a commission of the International Society for Photogrammetry and Remote Sensing (ISPRS)

(Torlegård *et al.*, 1986). The mean (average) slope angles are estimated as being 6°, 15° and 7° from photogrammetrically measured contours, for Uppland, Sohnstetten and Spitze respectively. The accuracies of measured data in terms of standard deviation for these areas are estimated as 0·67m, 0·16m and 0·08m, respectively. The wavelengths for these three areas are estimated as being 470m (by equation (22c)) for Uppland, 214m (the width of the area checked) for Sohnstetten and 300m (the width of the area checked) for Spitze. Also in the vicinity of Spitze, there are two steep slopes along both sides of roads. The values of these breaklines vary from 3m to 0·5m, with an average of 1·25m, thus $E_b = 1{\cdot}25$m.

Using these estimates for equations (22a) and (22b), theoretical predictions can then be made. A comparison of these predictions with the results obtained from practical tests is shown in Table I. It can clearly be seen from this table that some areas had over-estimated values for predicted accuracy, while others were underestimated. However, in general, the discrepancies are within expectation since it has been found that the accuracies for the two grids with the same interval, but with an offset of the origins or a different orientation could also be quite different. For example, for the Sohnstetten area, the largest difference of standard deviations for two 56·56m grids is 0·26m. Thus a value of 0·18m for the largest difference between the predicted value and tested value for the 56·56m grids is not surprising.

TABLE I. Comparison of predicted accuracy with test results.

Test areas	Grid interval (m)	*Grid data* Predicted (m)	Tested (m)	Difference (m)	*Composite data* Predicted (m)	Tested (m)	Difference (m)
Uppland	28·28	0·54	0·63	−0·09	0·51	0·59	−0·08
	40	0·64	0·76	−0·13	0·56	0·66	−0·10
	56·56	0·85	0·93	−0·08	0·66	0·70	−0·04
	80	1·24	1·18	0·06	0·81	0·80	0·01
Sohnstetten	20	0·63	0·56	0·07	0·45	0·43	0·02
	28·28	0·97	0·87	0·10	0·63	0·56	0·07
	40	1·56	1·45	0·11	0·87	0·78	0·09
	56·56	2·58	2·40	0·18	1·23	1·08	0·15
Spitze	10	0·17	0·21	−0·04	0·12	0·16	−0·04
	14·14	0·25	0·28	−0·03	0·15	0·17	−0·02
	20	0·38	0·35	0·03	0·20	0·18	0·02

Predict denotes the predicted accuracy values; *Tested* denotes the experimental accuracy values; and *Difference* denotes the differences between predicted and tested values, all in terms of σ.

From the experimental evaluation carried out above, it can be seen that these two accuracy models can produce reasonable prediction. It means that, in practice, they are good approximations. For theoretical evaluation, Meyer (1985) suggests that the following can be used as standards: (a) accuracy; (b) descriptive realism; (c) precision; (d) robustness; (e) generality; and (f) fruitfulness. To these six, Li (1990) added another criterion: (g) simplicity. Using these seven criteria, the models presented in this paper can be judged from a theoretical standpoint.

CONCLUDING REMARKS

In this paper, two mathematical models have been presented of the accuracy of a DTM surface, both linearly constructed from grid data only and from grid data with additional feature specific data. The surface accuracy can be simply expressed by a very general formula as follows:

$$\sigma^2_{Surf} = K_1\,\sigma^2_{nod} + K_2\,(1 + K_3 d)^2\,(d\tan\alpha)^2 \qquad (24)$$

where K_1 is a constant, approximately equal to 4/9, K_2 is also a constant (approximately equal to 5/768) depending on the characteristics of the terrain surface, and K_3 is also a constant which is equal to zero for composite data and for grid data only is approximately equal to $4/\lambda$ or (perimeter/area) of the lowest contour, d being the grid interval and α the average slope.

The main advantage of these models is that they offer a simple mathematical expression for surface accuracy, which is an analogue of a conventional map accuracy model and is thus very convenient for practical use, in addition to reliability of prediction. These models may lead to the development of mathematical models of the accuracy of contours which are produced from DTMs, or even models of the accuracy of other DTM products.

It needs to be emphasized here that it is impossible for an accuracy model to produce absolutely accurate predictions and the models described in this paper are not exceptions. Experience gained by the author from experimental work seem to show that a value of 15 per cent for the difference between standard deviations obtained from two different grids with the same grid interval (but possibly with an offset or with different orientation) is well within expectation and 20 per cent is even possible in some cases. Such a value might also be applied in case of model evaluation.

It should also be noted here that the models described in this paper are subject to many limitations. For example, the models have the restrictions that they only consider (a) square grid data (with and without additional feature specific data); (b) a linear surface and (c) the direct construction method. Further limitations include: (d) only a very rough estimate for the value of K has been given; (e) the orientation and location of grids are not considered; (f) the models are not applicable in the case of very large grid interval (for example $d \geq \lambda$).

The two models appear to be promising. However, due to the limitations mentioned above, a systematic evaluation would be desirable to check their applicability in practice. Especially, it would be beneficial to make some modification to K and $P(r)$ in order to fit the characteristics of a particular type of terrain surface. Further work in this respect may help the OEEPE to set a DTM standard, which is the objective of an OEEPE project.

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Résumé

On présente dans cet article deux modèles mathématiques sur la précision des modèles numériques du terrain (MNT) que l'on a établis linéairement à partir de données réparties selon une grille carrée, l'un, en tenant compte de données auxiliaires spécifiques sur la surface, l'autre sans.

On exprime, dans ces modèles, la précision sur la surface du MNT par une fonction de quelques paramètres tels que l'angle de pente, le pas de la grille ou la précision des données initiales, ce qui est tout à fait familier aux practiciens de la photogrammétrie et des autres disciplines liées à la cartographie. La forme de ces modèles est analogue à celle des formules habituelles concernant la précision des courbes de niveau des cartes; enfin on évalue ces modèles sur les résultats d'essais expérimentaux.

Zusammenfassung

Im Artikel werden 2 mathematische Modelle für die Genauigkeit von Oberflächen digitaler Geländemodelle (DTM) dargestellt, die linear aus quadratischen Rasterdaten, und zwar mit und ohne zusätzliche Geländedaten erzeugt wurden. Bei diesen Modellen wird die Genauigkeit der DTM-Oberflächen als Funktion weniger Parameter ausgedrückt, wie z.B. der Geländeneigung, dem Rasterabstand oder der Genauigkeit der Rohdaten, die den Praktikern der Photogrammetrie und weiteren mit der Kartenherstellung verbundenen Disziplinen vertraut sind. Diese Modelle ähneln in der Form der konventionellen Formel für die Genauigkeit von Höhenlinienkarten, und sie werden mit experimentellen Versuchsergebnissen verglichen.