Mathematical Basis for Non-Spherical Celestial Bodies Maps

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Abstract

The question about the exact mathematical basis for celestial bodies has not been answered yet. For example, investigations conducted by Canadian scientists on asteroid mapping have as a main goal the real models of body shape. During the last 40 years, a sphere has been used as the reference surface for mapping celestial bodies in the Solar System. Now is the time to discuss the use of an ellipsoid instead of a sphere for the Moon and Mars. Apart from terrestrial group planets and their moons, it can also be used for mapping asteroids and comet nuclei.

This paper will discuss the mathematical models for non-spherical celestial bodies, including the reference surface model and the map projection theory. Particularly the discussion will be conducted about the use trend surface, perspective azimuthal projections, pseudo-cylindrical and modified conic projections.

1 Introduction

The mapping of Solar System bodies began 40 years ago when the Soviet interplanetary station Luna-3 had photographed the other side of the Moon and transmitted the image of the lunar surface to the Earth for the first time. Since then, as a result of spacecraft missions to terrestrial planets and their satellites, to the giant planets with their numerous environments and to comets and asteroids, surface maps of many of the planets have been produced in various scales (Bugayevsky at al. 1992). These planets include Mercury, Venus, Mars, the Moon, Phobos, and Deimos, the Galileo satellites of Jupiter (Io, Europe, Ganimede, Callisto), the nucleus of Halley's comet, asteroids Gaspra, Ida, Matilda and so on.

For such mapping, the critical problem is the determination of a reference model. During the last 40 years, a sphere has been used as the reference surface for mapping celestial bodies in the Solar Systems. Now is the time to discuss the use of an ellipsoid instead of a sphere for the Moon and Mars. This paper will discuss the mathematical models for non-spherical celestial bodies, including the reference surface model and map projection theory. Particularly the discussion will be conducted about the use trend surface, perspective azimuthal projections, pseudo-cylindrical and modified conic projections.

2 Mapping Celestial Bodies: An Overview

With the acquisition of new data from planets in the Solar System, more precise knowledge of the form and size of these planets has been acquired. Such data have been used for the determination of reference surfaces.

For Mars (similar to the Earth), an ellipsoid with a small polar compression has been applied. Its parameters are as follows: *R polar = 3376.3 km, R equator = 3393.4 km*. Then for the first time, a triaxial ellipsoid with semi-axes of *13.5 km, 10.4 km and 9.6 km* respectively was tested for Phobos.

Maps have also been produced, using a conformal cylindrical projection from the globe in the form of triaxial ellipsoid (Bugayevsky 1971; Bugayevsky 1987, MIIGAiK&MGU 1988). The sphere was considered as the first approximation for the Moon. The character of the equi-potential surfaces of the Moon was studied and arguments regarding the benefit of using a triaxial ellipsoid

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as the second order approximation to the moon were also raised. Particularly, a decrease in geoid height differences was revealed when an ellipsoid is used as the reference surface for the Moon compared with height differences when a sphere is used as the reference surface. It was discovered that average deviations of the Moon's geoid heights from the ellipsoid are approximately 5 times more than the deviations of the Earth's geoid from the Moon's geoid. The question regarding whether the centre of the reference surface for the Moon should coincide with the mass centre of the Moon also needed to be decided.

Nowadays, the replacement of the sphere by an ellipsoid as a reference surface for the Moon has become a matter of some urgency, because of the new data transferred by the Clementine satellite (1994) and later confirmed and specified by the Lunar Prospector satellite (1998). The presence of water in the craters of the northern and southern polar areas essentially will lead to the consecutive exploration of the Moon, the establishment of a permanent working base there and also the utilisation of the Moon's natural resources (helium 3 minerals etc.). Such purposeful exploration of the Moon will require the creation of new, but more precise maps and the development of optimum cartographical projections both for the Moon as a whole and for different local regions.

Apart from the planets and their satellites, issues on asteroid's mapping also deserve serious attention. Photography of these bodies, begun with satellite Galileo and followed by the NEAR project (Near Asteroids Rendevous), delivered a large amount of data for mapping bodies with essentially irregular forms. Currently, a project called "New Millennium", which is wholly devoted to asteroid and comet research, is also being developed. Interest in asteroids is stimulated by the prospect of producing mineral raw materials. The realisation of this project is planned for 2015-2020. In Canada, scientists have also conducted investigations into the shape such bodies.

3 Issues on Map Projections for Non-spherical Celestial Bodies

From the point of view of mathematical cartographers, in order to produce maps of the surfaces of various celestial bodies, two issues need to be tackled:

a) The development of the theory and methods for projections of those bodies that are irregular in form but are approximated by regular surfaces (sphere, ellipsoid etc.) (Bugayevsky at al. 1992, Bugayevsky 1971).
b) The development of the theory and methods for projections of those bodies that are more irregular in form but are not approximated by regular surfaces (sphere, ellipsoid etc.).

Research on cartographic projection for the production of maps for such surfaces has already begun in Russia and other countries (Bugayevsky and Portnov 1984, Bugayevsky and Nyrtsov, 1998). The work of American and Canadian and many others is known (Stooke and Keller 1990). However, there are still some critical issues to be tackled:

Based on the data sets obtained from space missions, the following topics should be studied:

a) Further development of theory for description of the shape of non-spherical bodies, by using data from sounding and other sources.
b) Approximation of these surfaces by a sphere and the development of real surface height models concerning a sphere, i.e. construction and analysis of models of surface heights for ellipsoids and triaxial ellipsoids, compared to spheres in various coordinate systems.
c) Development of a common theory of cartographic projections for irregular surfaces and a theory of classifications of cartographic projections for non-spherical body maps with various territorial scopes (different scales).
d) Development of methods for the evaluation of a given cartographic projection, for the computation of a primary scale and for the determination of other characteristics of a projection.
e) Analysis of real surface images taken by various space photography systems, and the definition and analysis of their mathematical models; i.e. comparative analysis of the same cartographic projection for presentation of those real images, taken by various space photography systems in various coordinate systems.

f) Calculation of optimum vertical scales for real surface image construction.
g) Development of ellipsoid and triaxial ellipsoid projections, study of the characteristics of distortions with height models and projection formulae of the sphere model mentioned before, and comparative analysis for the production of maps with various territorial scopes (scales).
h) Analysis of cartographic projections in terms of convenience for studying the form and morphology of those non-spherical bodies.

4 Alternative Map Projections for Non-spherical Celestial Bodies

In the previous section was a brief discussion of some tasks to be carried out in the future. In this section some alternative map projections will be examined.

So far, cartographic projections developed for non-spherical bodies mainly include, perspective azimuthal projections, whose formulae contain three parameters: the latitude, longitude and differences of height points of a surface (altitude).

$$X = f_1 (z, a, h); \quad Y = f_2 (z, a, h) \tag{1}$$

For example, the formulae for modified orthographic and stereographic projections can be written in a more concrete form as follows:

Orthographic projections:

$$X=(R+h) \times \sin z \times \cos a; \quad Y=(R+h) \times \sin z \times \sin a \tag{2}$$

Stereographic projections:

$$X=2 \times (R+h) \times tg\, z/2 \times \cos a; \quad Y=2 \times (R+h) \times tg\, z/2 \times \sin a \tag{3}$$

Where, z is in a-polar spherical coordinates. It is also possible to use other projections, such as

Lambert projection:

$$X=2 \times (R+h) \times \sin z/2 \times \cos a; \quad Y=2 \times (R+h) \times \sin z/2 \times \sin a. \tag{4}$$

Postel projection:

$$X=(R+h) \times z \times \cos a, \quad Y=(R+h) \times z \times \sin a. \tag{5}$$

Applications of these projections are convenient for the study of the form and morphology of bodies up to hemispheres. It is possible to use pseudo-conic and modified conic projections for same purpose.

If required to study body surfaces of the bodies in planetary scale, it is expedient to use cylindrical, pseudo-cylindrical and polyconic projections, as follows:

Cylindrical projection:

$$X=(R+h) \times f (\varphi) , Y=(R+h) \times \lambda \tag{6}$$

or

$$X=(R+h) \times f_1(\varphi) , Y=(R+h) \times f_2 (\lambda). \tag{7}$$

Pseudo-cylindrical projection:

$$X=(R+h) \times f_1(\varphi), Y=(R+h) \times f_2(\varphi,\lambda). \tag{8}$$

Polyconic projection:

$$X=(R+h) \times [q(\varphi) - \rho(\varphi) \times \cos\delta(\varphi,\lambda)], \quad Y=(R+h) \times [\rho(\varphi) \times \sin\delta(\varphi,\lambda)]. \tag{9}$$

It is also possible to use projections such as Aitov and Vinkel's.

In all the projections specified above, the displacement of points on a projection will be in both abscises and ordinate directions. Sometimes, it might be necessary to research those images, on which the influence of a relief is only in abscises or ordinate direction. If so, then general equations of the cylindrical projections of real surfaces can be written as follows:

$$X=(R+h) \times f(\varphi), \quad Y=c \times \lambda \quad \text{or } Y=f(\lambda). \qquad (10)$$

To generate maps of irregular body’s surfaces, the choice of the vertical scale for the image construction is very important. The larger the vertical scale is, the clearer is the influence of altitude values on the displacement of a projection. The features of the surface will be clearer too. Simultaneous definition and study of the distribution of the distortion magnitude with a given projection is a more difficult task.

The definition of the distortion value by traditional methods of projection becomes impossible because the surfaces are not smooth. Their differences, -- from smooth surfaces of spheres or ellipsoids, -- as the image vertical scale increases.

Therefore, it is necessary to perform image smoothing, coordinate transformations, but under the condition that, in the developed projections, the opportunities for studying the features and morphology are basically kept. For these purposes, the known methods of linear and non-linear smoothing, as well as smoothing methods based on approximation etc. can be used.

For the same reason, it is necessary to develop trend models to transform the basic features of real surfaces and at the same time give the opportunity to define the primary scale, distortion values and other projection characteristics. The known approximation methods for a relief such as polynomials and splines can be used for constructing trend surfaces (Bugayevsky, and Vachrameeva 1992; Snyder 1995).

At the present time, various kinds of space photography systems (frame, scanners, radar-location etc.) are available to take images of the real surface. The images obtained represent projections of surfaces received by physical methods.

In this case, the relief on the surface of these bodies has direct influences on the image formation by these image systems. There are questions about selecting study methods for these images, developing methods for their mathematical description as well as studying the features of real surfaces and for deciding which photogrammetric and cartographic tasks to use.

There must be a comparative analysis of all developed methods, the results of the research, the mathematical models of height and trend surface models etc.

For the development of a technique for creating such models, it is possible to use ellipsoid and triaxial ellipsoid surfaces as real surfaces and to determine the differences of heights on these surfaces compare to a sphere surface.

If such research on an ellipsoid projection is basically of only theoretical interest, the adoption of triaxial ellipsoid specified projections has equal theoretical and practical importance.

5 Conclusions

In the present paper, the basic direction of the researches is briefly described. The completion of these tasks will form the theoretical basis for cartographic projections of real surfaces for the generation of maps for various assignments, contents and territorial scope (scale).

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